where the integrand J is the same as the integrand in Eq. (10).

Equations (10) and (11) have been numerically integrated and the results are plotted in Fig. 1. Also shown for comparison is the sketched curve given by Browand for $n/n_0 = 0.95$. Along the centerline $r = 0$ an expression for the density in terms of elementary functions can be given, since then $a = Z$ and $b = -Z$, and so Eq. (10) reduces to

$$n/n_0 = Z(1 + Z^2)^{-1/2} \exp[-V_0^2/(1 + Z^2)] \qquad (12)$$

As just mentioned, the expression for the density given by Gurevich[2] is approximate. Unfortunately, Ref. 2 contains no statement of the approximation. However, in another paper Gurevich[3] considers the same problem and assumes that the molecules possess no thermal component of velocity in the z direction.‡ This means that in Eq. (5) the c_z that appears in the argument of the sign function is replaced by c_0, but the exponential is unchanged. This is necessary so that the freestream density at $z = 0$ can be denoted by n_0. The c_z integration can then be done immediately. Since the argument of the sign function is just the characteristic of the differential equation, it represents the trajectories of the molecules from the plane $z = 0$ into the region of interest ($z > 0$) and corresponds to the assumption of constant velocity c_0 in the z direction. One can then proceed to carry out the remaining two integrals by a method analogous to that just used, and by making use of a transformation similar to that given by Kornowski[4] in the study of the nonsteady expansion of a gas into an initially evacuated, infinitely long cylinder. The integral for the density given by Gurevich [Eq. (9), Ref. 3] is correct based on the assumption just mentioned and has been tabulated by Brinkley[5] and by Masters.[6]

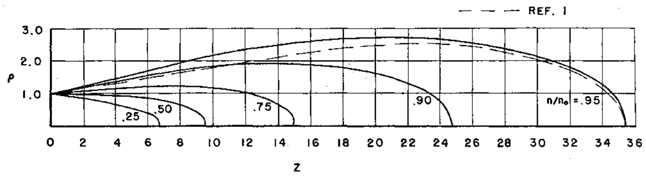

Fig. 1 Constant density lines for $V_0 = 8$.

Gurevich's expression for centerline density is

$$n/n_0 = \exp[-(V_0/Z)^2] \qquad (13)$$

and comparing this with Eq. (12), it is clear that the two expressions are nearly the same, except for small values of Z. This explains the relatively good agreement between the exact calculations shown in Fig. 1 and Gurevich's approximate results. A complete discussion of this and related problems will soon be available as a Purdue University report.

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Temperature Distribution in a Case-Bonded Cylindrical Rocket Assembly

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Introduction

When a case bonded propellant grain is subjected to changes in thermal environment, bond stresses develop as a result of a differential thermal expansion between the casing and the propellant grain. The complexity of the bond stress equation depends upon the prescribed temperature condition. In the case-bonded propellant grain problem, the transient temperature expression is a complicated function of space and time so that the corresponding bond stress equation is extremely cumbersome. To simplify the bond stress expression for use in engineering analyses, studies were recently presented[1, 2] in which the temperature of the propellant grain was chosen to be either equal to the original equilibrium temperature or equal to the temperature of the new environment. These simple temperature conditions provide simple bond stress equations which correspond to the worst stress conditions in an elastic material.[3]

The forementioned simplifying conditions are no longer satisfactory for a viscoelastic analysis of the problem, because the resulting stresses are both time and space dependent. It is suggested that the difficulties now be overcome by assuming that the temperature of the propellant grain is independent of the radial coordinate but varies only with the time coordinate. Justification for the proposed temperature variation will be discussed in the present paper. The associated viscoelastic solution has been presented elsewhere.[4]

Temperature Distribution

The cross section of a typical solid propellant rocket assembly is shown in Fig. 1. It may have either a solid propellant grain of external radius b or a hollow propellant grain of internal radius a and external radius b, surrounded by either a metallic or plastic casing that is physically thin in comparison to its external radius c. From a heat-transfer point of view, the outer surface of the assembly has a convective heat-transfer coefficient h_0, and the bond surface between the propellant grain and the casing has a thermal resistance h_i.

A general equation for the temperature distribution within the propellant grain is, practically, difficult to write under arbitrary boundary conditions because of the complexity of the algebra involved. If, however, the range of ambient temperature variation and heat-transfer parameters is restricted, a solution may be found which lends itself to further use in viscoelastic analysis.

From considerations of elementary heat transfer, one can write an expression for the temperature-time distribution in a cylindrical grain which is initially at a uniform temperature T°, but whose temperature is altered when the ambient temperature changes to zero at time $t = 0$, namely,[5]

$$T(r,t) = 2T^\circ \sum_{n=1}^{\infty} \left[\frac{J_0(\lambda_n r)J_1(\lambda_n b)}{J_0^2(\lambda_n b) + J_1^2(\lambda_n b)}\right] \frac{e^{-\alpha t \lambda_n^2}}{\lambda_n b} \qquad (1)$$

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where λ_n are the roots of

$$\lambda_n b \left[\frac{J_1(\lambda_n b)}{J_0(\lambda_n b)}\right] - N_{Bi} = 0 \qquad (2)$$

and N_{Bi} is the Biot modulus, defined by the expression

$$N_{Bi} = Hb/k_p \qquad (3)$$

$J_0(\lambda_n b)$ and $J_1(\lambda_n b)$ are the Bessel functions of zero order and the first order respectively; α is the thermal diffusivity of the propellant grain equal to $k_p/\rho c_p$, where k_p is its thermal conductivity, c_p is its specific heat, and ρ is its density.

The total influence of the casing is expressed by a single surface coefficient H, where

$$1/H = (1/h_0) + (1/h_c) + (1/h_i) \qquad (4)$$

and the equivalent film coefficient

$$h_c = \frac{k_c}{b\ln(c/b)} = \frac{k_c}{b\{\ln[1 + (d/b)]\}} = \frac{k_c}{d} \qquad (5)$$

where k_c is the thermal conductivity of the casing whose thickness d is small with respect to the radius b. In the limit, when the metallic casing is very thin, h_c is much greater than h_0, and may also be much larger than h_i, so that the thermal resistance offered by the casing is relatively small. One may then justifiably omit the term $1/h_c$ in Eq. (4), and treat the casing analytically as a region where there exists a separation of the surface film h_0 and the bond film h_i. Under this condition, the temperature of the casing T_c, may be expressed in terms of the ambient temperature T_0 and the temperature at the outer surface of the propellant T_b, as

$$T_c = \frac{h_0 T_0 + h_i T_b}{h_0 + h_i} \qquad (6)$$

Unfortunately, Eq. (1) for the temperature of the propellant grain is not as simple. It converges so slowly that its use in a viscoelastic stress analysis becomes exceedingly laborious. Even though charts based on Eq. (1) are available,[6] the exact temperature equation does not easily lend itself to further analytical use. However, the charts show that at any given time t, the temperature variation within the propellant grain decreases as the Biot modulus N_{Bi} decreases. Thus, at any prescribed time t, a uniform temperature distribution is a reasonable approximation of the temperature distribution throughout the propellant grain as long as the Biot number is small.

Charts based on Eq. (1) also can be used to determine the variation between an assumed uniform temperature and the actual temperature distribution. An examination shows for example, that for Biot moduli between 0.05 and 2.0, the ratio $(T_a - T_b)/T_a$, where T_a is the temperature of the center and T_b is temperature of the surface of the propellant grain, varies in the manner as shown in Table 1. Obviously, then, the radial variation of temperature is indeed small for small Biot numbers.

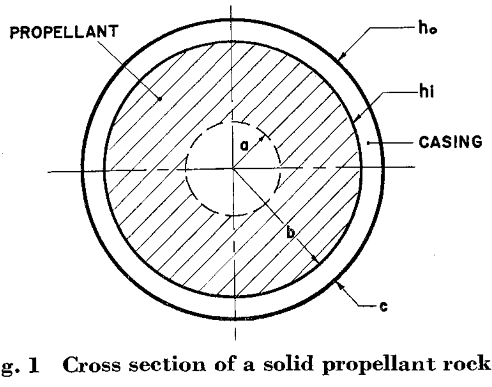

Fig. 1 Cross section of a solid propellant rocket.

Table 1

N_{Bi}	2.0	1.00	0.50	0.20	0.10	0.050
$(T_a - T_b)/T_a$	0.5	0.36	0.21	0.09	0.05	0.025

If, as suggested, we assume that the temperature of a propellant grain is independent of the radial coordinate, the rate of heat transfer per unit length from a propellant grain with film coefficient H may be written[5]

$$\dot{Q} = \pi b^2 c_p (dT/dt) = 2\pi\, bHT \qquad (7)$$

when the ambient temperature is suddenly changed to zero from an equilibrium temperature $T°$. The temperature of the propellant grain is given by the solution of Eq. (7) as

$$T = T_0 e^{-(2Ht/b\rho c_p)} \qquad (8)$$

where c_p is the specific heat and ρ is the density of the propellant grain.

Temperature-Time Relations with Small Biot Moduli

Let us consider in greater detail the temperature-time relation of the propellant and casing when a solid propellant rocket is subjected to a step-like change in ambient temperature. If the equivalent film coefficient h_c is much larger than the convective heat-transfer coefficient h_0 and the thermal resistance h_i at the bond surface between the propellant grain and its casing, the temperature of the casing is uniform. Under these conditions the propellant and casing temperature-time relations are given by Eqs. (8) and (6) respectively, with $T_b = T$.

For a fixed value of the overall coefficient H, limiting cases of $h_0 \gg h_i$, and $h_i \gg h_0$ give rise, respectively, to situations where the casing takes on the ambient temperature immediately, or responds less rapidly with the bulk of the propellant. The first occurs when there is an effective insulating layer between the casing and propellent, and the second, when a direct bond between casing and propellent exists.

It is evident from Eq. (6) that when $h_0 \gg h_i$, the propellant temperature will be much less than the casing temperature during the initial time interval. Not only will this situation occur when an insulating film of sufficient thermal resistance exists between the casing and propellant, but it will also occur when the propellant has a low thermal conductivity. In either situation, the propellant grain responds slowly enough to any ambient temperature change so that, for all practical purposes, the temperature remains constant long enough for viscoelastic stress changes to take place. Thus, if the viscoelastic relaxation time is sufficiently short when compared to the propellant temperature time constant as expressed in Eq. (8), the viscoelastic stress problem may be investigated on the assumption that the temperature of the propellant remains unchanged while major stress variations occur.

Concluding Remarks

Equations have been presented for the exact temperature distribution in an axially symmetric case bonded propellant grain when the ambient temperature is suddenly changed from a state of equilibrium. They were found to be lengthy, and their use in a viscoelastic stress study would obviously lead to considerable practical difficulties.

It was found that when the Biot modulus is small, the temperature variation within the propellant in the radial direction is also relatively small. It is then reasonable to assume in an engineering calculation that the temperature of the propellant grain is uniform and varies with the time coordinate only. Under the aforementioned conditions it is shown that the propellant temperature varies exponentially with time at a rate which depends upon the geometry and the physical properties of the propellant grain, as well as the over-all surface coefficient H.

Use of the uniform, time dependent temperature distribution in another analysis[4] has lead to relatively simple expressions for the bond stress with a viscoelastic propellant.

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Comment on "Insulation Requirements for Long-Time Low-Heat Rate Environments"

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In a previous note,[1] the authors discussed the problem of insulation requirements for long-time low-heat rate environments. With several simplifying assumptions, the problem was reduced to that of heat conduction in a slab. It was stated that the solution to the diffusion equation for the conditions of interest could not be found in standard heat conduction tests, and a solution was presented. It has subsequently been brought to the authors' attention that the solution presented in Ref. 1 is available in standard texts, such as Ref. 2. A solution to the same problem using Laplace transforms is also presented in Ref. 3. The latter solution is of a more convenient form than that obtained using Fourier series and is shown in Fig. 1. It is noted that the required insulation weight per unit area appears only in the ordinate. Thus, the insulation weight per unit area may be obtained directly without the iteration process demanded by the figure in the original note.

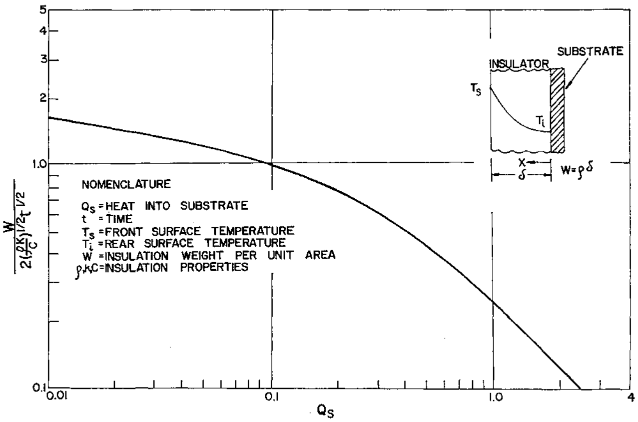

Fig. 1 Heat transmitted to rear surface of slab whose surfaces are maintained at constant temperatures.

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Correction and Addition to "Survey of Current Literature on Satellite Lifetimes"

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It has recently been brought to the attention of this author that Table 2 of Ref. 1 contains erroneous altitude data on the satellites noted and incorrect lifetime estimates.† Corrected perigee and apogee altitudes are (from top to bottom)

$$h_p = 95, 86, 123, 118, 120, 90, 104 \text{ naut miles}$$

$$h_a = 797, 526, 190, 391, 466, 477, 916 \text{ naut miles}$$

The estimated lifetimes associated with the above altitudes, however, have not been computed at this time.

Equations (66) and (67), which were hailed as rather general expressions for obtaining the number of revolutions and the time to decay, contain both the final radius and eccentricity of the orbit. In order to evaluate these expressions, an equation relating these quantities is needed. Reference 2 shows that the appropriate relationship is

$$\frac{e_1}{1+e_1}\rho_{p_1} j_1\left(\frac{\beta r_{p_1} e_1}{1+e_1}\right) = \frac{e_0}{1+e_0}\rho_{p_0} j_1\left(\frac{\beta r_{p_0} e_0}{1+e_0}\right) \tag{1}$$

where the terms are defined in Ref. 1.

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† The lifetimes shown in the table do not correspond to the satellite data listed.

Optimization of System Reliability

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While conducting an analysis of a space power system, this author came upon a criterion for optimizing system reliability which seemingly presented a new insight into

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